One-loop dimensional reduction of the linear σ model.

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Abstract

We perform the dimensional reduction of the linear σ model at one-loop level. The effective potential of the reduced theory obtained from the integration over the nonzero Matsubara frequencies is exhibited. Thermal mass and coupling constant renormalization constants are given, as well as the thermal renormalization group equation which controls the dependence of the counterterms on the temperature. We also recover, for the reduced theory, the vacuum unstability of the model for large N.

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1 Introduction

The $O(N)$ invariant linear σ model is particularly interesting in the study of high-density matter. For instance, when $N = 4$ it may be taken as a model for the low energy dynamics of QCD, and also it forms one of the basic theoretical tools for the description of the pion-condensed state of neutron stars matter. At low temperatures, the model presents a spontaneous breakdown of the $O(N)$ rotational invariance. One though expects a second order phase transition at some temperature, since the linear σ model is in the same class of universality of the classical N component Heisenberg ferromagnet. In the context of QCD, high temperature expansion of static (i.e. having external legs corresponding to the zero Matsubara mode) two-point functions have been obtained by Gale and Kapusta [1]. However, it is, in general, very difficult to carry out these calculations beyond the leading order in the loopwise expansion. On general grounds, in order to circunvect those computational difficulties we relieve, among others, two different methods of studying the high-temperature properties of a quantum field theory. The first one is based on a resumed perturbation theory where the introduction of a generalized effective action $\Gamma(\phi)$, which depends not only on the field ϕ but also on a composite operator $G(x, y)$, allows one to sum up an infinite set of higher order diagrams [2]. The second one is the so called dimensional reduction [3], which is based in the Appelquist-Carazzone decoupling theorem [4]. This method allows one to relate very hot thermal correlation functions in a D-dimensional space to zero temperature correlation functions in a space of dimension $D - 1$, giving a "dimensionally reduced" effective theory. In other words, hot field theories in $D = d + 1$ dimensions are related to zero temperature

effective field theories, in d dimensions, at the price that the renormalized parameters of this new theory have a dependence on the temperature [5][6]. Those must be considered at the limit of infinite temperature as new bare parameters subject to a second renormalization procedure so that the reduced theory makes sense.

The dynamical study of thermal field theories is based on the computation of the partition function $Z = tre^{(-\beta H)}$, where H is the Hamiltonian and β^{-1} is the temperature. The fact that in QCD and in other models of quantum field theory at finite temperature we have dramatic long range correlations when any of the boson masses approaches zero is one of the most important yet unsolved problems in thermal quantum field theory. For a recent discussion on this subject see for instance [7]. In general, for thermal, or even non-thermal, field theories there is not a procedure equivalent to the Bloch-Nordsieck mechanism in QED, for the computation of the physically relevant quantities. In QED, this procedure changes the definition of the asymptotic states in order to include all soft photons, giving infrared finite coss-sections. In general quantum field theories the infrared divergences causes the breakdown of perturbative calculations at some loop order and one would be left only with the hope of finding non-perturbative techniques to deal with theories like, for example, QCD. As far as we know, we are still far from having a satisfactory answer to this problem. Even if we use lattice simulations to compute correlation functions for static observables in hot QCD, we see that the computational resources that are required increase rapidly with β^{-1}. It is perhaps interesting to recall that in the case of zero temperature QCD, the non-cancellation of infrared divergences is expected to be closely related to the confinement

mechanism [8] [9] [10]. On the other side, for finite temperature QCD, it is expected that, at very high temperatures, QCD undergoes a color deconfinement phase transition where hadrons would melt in a quark-gluon plasma. The connection between these two aspects, the infrared behavior of QCD and the deconfining phase transition at high temperatures has been investigated, by many authors, since the pioneering work of Kalashnikov [11]. An interesting question would be to investigate whether this phase transition (the existence of some critical temperature) appears for hot QCD in the context of Matsubara formalism. Then it would be natural to expect some features of the $D-1$ dimensional reduced QCD to be those of a quark-gluon plasma in this space dimension as well.

At this point dimensional reduction is considered as a nice tool to investigate the above problems. First of all, the leading infrared behavior of four dimensional QCD at high temperature is governed by the static (zero Matsubara frequency) sector, obtained by integrating out the nonstatic modes, leaving an effective three dimensional field theory. Indeed, the introduction of temperature give us propagators of the kind

$$\Delta(\mathbf{p}, \omega_n) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}, \tag{1}$$

where the $\omega_n' s$ are the well known Matsubara frequencies $\omega_n = 2\pi n/\beta$ for bosons and $\omega_n = 2\pi(n + \frac{1}{2})/\beta$ for fermions. Infrared divergences of massless ($m = 0$) thermal field theories arise solely from the blowing up of the ($\omega_n = 0$) terms in the sum over boson energies as $\mathbf{p} \rightarrow 0$. In this sense we will construct an effective three-dimensional theory with purely $\omega_n = 0$ terms and all $\omega_n \neq 0$ absorbed into the physical parameters of the effective hamiltonian. The only modes

that propagates over distances larger than β are the bosonic $n = 0$ modes, and they will be the only responsible for infrared divergences of the theory. On the other side, in the high temperature limit, new interaction terms will arise, all with temperature dependent coeficients. Although it is not known, in general, very much about the exact form of the temperature dependence of the parameters in the effective theory, this dependence does not have any connection with the infrared behavior of the model and the parameters are expected to be analytical functions of the temperature. Fortunately, this is all the information we need for our purposes. There are other ways of obtaining a dimensionally reduced theory without the necessity of integrating out degrees of freedom [12], however, to the order we are concerned, all those methods give the same answers.

Concerning the color deconfinement there are still unanswered questions. Does QCD have a finite temperature phase transition? What is its order and what would be its critical temperature? As pointed out, for instance, by Kalashnikov, the quark-gluon plasma should exist in reason of the smallness of the effective interaction parameter at suficiently high temperatures, which allows to use perturbative tools to study some properties of the plasma. The absence of confinement in QCD as $T \to \infty$ has been proved by Polyakov [13] and Susskind [14], which implies that a quark-gluon plasma must be formed when the temperature is large enough. Clearly the deconfining phase transition must exist in order to explain the rigorous results of Polyakov and Susskind. Phenomenologically such a phase transition is expected and even in a rough sense "exhibited" (see for instance [15]). However from a theoretical point of view it has not yet been possible to put both the quark-gluon plasma and the condensed hadronic phase within a single framework. Con-

densation of the quark-gluon plasma into hadronic matter, which must take place at temperatures about $200Mev$, is not yet clear and we have at our disposal only qualitative or pure phenomenological considerations to explain such a phenomenon. We believe that dimensional reduction is a useful tool for understanding the very high temperature regime of quantum fields. In particular we expect from the study of simple models as the one treated in this paper, to learn about the mechanism of passing from low to very high temperature states, with the hope that this learning could be later applied in the study of more realistic models as QCD.

Our choice of the linear σ model [16] was based in the fact that, as previously said, when $N = 4$ we have a model for the low-energy dynamics of QCD [17]. At the same time, the model has other features we are interested in, like spontaneous breakdown [18] of $O(N)$ rotational invariance by the vacuum and a symmetry-restoring phase at finite temperature.

The paper is organized as follows. In section II we briefly discuss the linear σ model at finite temperature. In section III, the effective potential of the reduced theory is obtained as well as the counterterms that allow dimensional reduction. Conclusions are given in section IV . In this paper we use $\frac{h}{2\pi} = c = 1$.

2 The linear σ model at finite temperature

We are interested in studying the behavior of a model of an N-component real massive scalar field $\phi = \{\phi^a(x)\}$ with the usual $\lambda(\phi^a\phi^a)^2$ self-interaction, defined in a static spacetime in the limit of high temperature. Since the manifold is static, there is a global timelike Killing vector

field orthogonal to the spacelike sections. Due to this fact, energy and thermal equilibrium have a precise meaning.

The partition function can be represented as a functional integral over fields $\phi(t, \mathbf{x})$ defined on a time interval 0 to $-i\beta$. It is usual to change variables from t to imaginary time $\tau = it$, which implies that the partition function is given in terms of the Euclidean functional integral

$$Z = \int \mathcal{D}\phi e^{-S_E(\phi(\tau,\mathbf{x}))}, \tag{2}$$

where

$$S_E = \int_0^\beta d\tau \int d^3\mathbf{x} \mathcal{L}(\phi(\tau, \mathbf{x})), \tag{3}$$

is the classical action. The Lagrangean density of the linear σ model is given by

$$\mathcal{L}(\phi^a(\tau, \mathbf{x})) = \frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{1}{2}\mu^2 \phi^a \phi^a + \frac{\lambda}{4}(\phi^a \phi^a)^2, \tag{4}$$

where we notice that the mass term has a minus sign to allow a spontaneous symmetry breaking. The trace in the partition function is implemented by imposing the following periodic boundary conditions on the bosonic fields

$$\phi^a(\tau, \mathbf{x}) = \phi^a(\tau + \beta, \mathbf{x}). \tag{5}$$

Because of this periodicity we may perform a Fourier expansion for the fields in the τ variable by writting

$$\phi^a(\tau, \mathbf{x}) = \beta^{-\frac{1}{2}} \sum_{n=-\infty}^{+\infty} \phi_n^a(\mathbf{x}) e^{i\omega_n \tau}, \tag{6}$$

so that, after integration over τ we have

$$S_E = \int d^3\mathbf{x} \left\{ \frac{1}{2} \sum_n (\partial_\mu \phi^a_n \partial_\mu \phi^a_{-n} - \mu^2 \phi^a_n \phi^a_{-n}) + \frac{\lambda T}{4} \sum_{n_1,n_2,n_3,n_4} \delta(n_1 + n_2 + n_3 + n_4) \phi^a_{n_1} \phi^a_{n_2} \phi^b_{n_3} \phi^b_{n_4} \right\}, \tag{7}$$

where we have made use the well known relation

$$\int_0^\beta d\tau e^{i(\sum_n \omega_n)\tau} = \beta\delta(\sum_n \omega_n). \tag{8}$$

Before explicitly computing the effective potential for this action we can already see that the parameters will have a dependence on the temperature. Here the temperature enters via the coupling λT, but we still have to take into account a contribution comming from the frequencies ω_n in the computation of non-static Feynman loops. In the next section we explicity obtain the one-loop effective potential of the reduced model.

3 The effective potential of the reduced model.

As we have pointed out in the introduction, the effective three-dimensional field theory is obtained from the integration of the $\omega_n \neq 0$ modes in the Euclidean functional integral given by eq.(2). This can be done by performing a semi-classical expansion of Z at one loop level [19]. For this we consider the fields configurations ϕ^a around a particular constant field Φ^a_0, the static part of the solution of a saddle-point equation for the original action given by eq.(3). We write $\phi^a \to \Phi^a_0 + \eta^a$,

so that, after performing the Gaussian integration over η we are lead to

$$Z = e^{-S_E(\Phi_0^a)} \det \left[\left. \frac{\delta^2 S}{\delta\phi^a \delta\phi^b} \right|_{\phi=\Phi_0} \right]^{-\frac{1}{2}} . \tag{9}$$

The determinant given by eq.(9) can be exactly calculated and gives the one loop effective potential in terms of Φ_0^a. From the action given by eq.(7) we get

$$\left. \frac{\delta^2 S}{\delta\phi^a \delta\phi^b} \right|_{\phi=\Phi_0} = -\sum_n [(\bigtriangledown^2 - \omega_n^2 + \mu^2) - \lambda T(\Phi_0^k \Phi_0^k \delta^{ab} + 2\Phi_0^a \Phi_0^b)]. \tag{10}$$

Since we have rotational invariance of the vacuum, one can choose a coordinate system, in the fields configuration space, in which $\Phi_0^a = (0, ..., \phi_0)$. Now, we define

$$m_i^2 = \begin{cases} \lambda T \phi_0^2 & \text{if } i = 1, ..., N-1 \\ 3\lambda T \phi_0^2 & \text{if } i = N \end{cases} , \tag{11}$$

so that

$$\left. \frac{\delta^2 S}{\delta\phi^a \delta\phi^b} \right|_{\phi=\Phi_0} = -\sum_n (\bigtriangledown^2 - \omega_n^2 + \mu^2 - m_i^2). \tag{12}$$

To proceed, we make use of the relation

$$Tr \ln \left(-\sum_n (\bigtriangledown^2 - \omega_n^2 + \mu^2 - m_i^2) \right) = (VT) \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln \left(\sum_n (\mathbf{k}^2 + \omega_n^2 - \mu^2 + m_i^2) \right), \tag{13}$$

where V is a volume element and the integral over $\mathbf{k}$ can be done with the aid of the identity [20]

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} \ln(\mathbf{k}^2 + m^2) = -\frac{\partial}{\partial s} \left[\int \frac{d^d\mathbf{k}}{(2\pi)^d} (\mathbf{k}^2 + m^2)^{-s} \right]_{s=0} . \tag{14}$$

In order to simplify the expressions we define the quantities

$$\begin{cases} a_i^2 = \omega_n^2 - \mu^2 + m_i^2 = (\frac{2\pi}{\beta})^2 (n^2 + \alpha_i^2) \\ \alpha_i^2 = (\frac{\beta}{2\pi})^2 (m_i^2 - \mu^2) \end{cases} \tag{15}$$

and split the sum over n as

$$\sum_n (n^2+\alpha_i^2)^{-s} = \alpha_i^{-2s} + 2\sum_{n=1}^{\infty}(n^2+\alpha_i^2)^{-s}. \tag{16}$$

The integral over $\mathbf{k}$, in eq.(14), is done using the well know result from dimensional regularization,

$$\int \frac{d^d\mathbf{k}}{(\mathbf{k}^2+a_i^2)} = \pi^{\frac{d}{2}}\frac{\Gamma(s-\frac{d}{2})}{\Gamma(s)}\frac{1}{(a_i^2)^{s-\frac{d}{2}}} \tag{17}$$

and the sum over the Matsubara frequencies is done using an analytical extension of the Hurwitz zeta-function [21]. This procedure gives for the determinant the result

$$\ln\det\left[\left.\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\right|_{\phi=\Phi_0}\right] = \frac{\partial}{\partial s}\left[\frac{\pi^2}{(2\pi)^3}\frac{\alpha_i^{4-2s}}{\Gamma(s)}\left(\frac{\beta}{2\pi}\right)^{2s-3}\left(\Gamma(s-2)+4\sum_{n=1}^{\infty}(\pi n\alpha_i)^{s-2}K_{s-2}(2\pi n\alpha_i)\right)\right]_{s=0}, \tag{18}$$

where K_ν is the modified Bessel function [22].

The exact one-loop renormalized effective potential for the thermal linear σ model can finally be written, with $m_\pi^2 = \lambda T\phi_0^2$ and $m_\sigma^2 = 3\lambda T\phi_0^2$ as

$$\begin{aligned}
V_{\text{eff}}(\phi_0^2) \;=\; & -\frac{1}{2}\mu^2\phi_0^2 + \frac{\lambda T}{4}\phi_0^4 + \\
& \beta\frac{\Gamma(-2)}{2^5\pi^2}\left[(N-1)(m_\pi^2-\mu^2)^2 + (m_\sigma^2-\mu^2)^2\right] - \\
& \frac{\beta^3}{3}\frac{\Psi(0)}{2\pi^2}\left[(N-1)(m_\pi^2-\mu^2)^2\int_1^\infty \frac{(t^2-1)^{\frac{3}{2}}}{e^{\beta(m_\pi^2-\mu^2)^{\frac{1}{2}}t}-1}dt + \right. \\
& \left.(m_\sigma^2-\mu^2)^2\int_1^\infty \frac{(t^2-1)^{\frac{3}{2}}}{e^{\beta(m_\sigma^2-\mu^2)^{\frac{1}{2}}t}-1}dt\right] + \\
& \frac{1}{2}\delta\mu^2\phi_0^2 + \frac{\delta\lambda T}{4}\phi_0^4,
\end{aligned} \tag{19}$$

where we have used the representation [22]

$$K_\nu(z) = \frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}dt, \tag{20}$$

for the modified Bessel function $K_\nu(z)$.

At this point it would be natural to start the search for the counterterms that suppress the ultraviolet divergences and make the theory finite when the limit of high-temperature is taken. However we know that it is more intuitive to read these quantities directly from the series expansion of the eq.(13) in powers of ϕ_0^2. In this case, instead of using eq.(14) we write

$$\ln\det\left[\left.\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\right|_{\phi=\Phi_0}\right] = \sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s}(m_i^2)^s\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_n\frac{1}{(\mathbf{k}^2+\omega_n^2-\mu^2)^s}, \tag{21}$$

and, by defining the quantity

$$\alpha^2 = (\frac{\beta}{2\pi})^2(-\mu^2), \tag{22}$$

we have, after integration over $\mathbf{k}$ and summation over n, the expression

$$\begin{aligned} V_{\text{eff}}(\phi_0) \;=\;& -\frac{1}{2}\mu^2\phi_0^2 + \frac{\lambda T}{4}\phi_0^4 + \\ & \sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s}\frac{1}{2^3\pi^{\frac{3}{2}}}\left[(N-1)m_\pi^{2s}+m_\sigma^{2s}\right](\frac{\beta}{2\pi})^{2s-3} \\ & \left\{\frac{\pi^{\frac{1}{2}}}{2\alpha^{2s-4}}\frac{1}{\Gamma(s)}\left[\Gamma(s-2)+4\sum_{n=1}^{\infty}(\pi n\alpha)^{s-2}K_{s-2}(2\pi n\alpha)\right]\right\} + \\ & \frac{1}{2}\delta\mu^2\phi_0^2 + \frac{\delta\lambda T}{4}\phi_0^4. \end{aligned} \tag{23}$$

As explicitly pointed out by Landsman [6], the possibility of dimensionally reducing a thermal field theory, depends on whether we are able to find thermal counterterms that cancel the divergent

contributions as $\beta \rightarrow 0$ comming from the integration of non-zero Matsubara frequencies. These temperature dependent counterterms should be controlled by a thermal renormalization group and fixed by the imposition of renormalization conditions on Feynman diagrams computed with the aid of some regularization scheme. In spite of regularization being purely an intermediary tool in the process of renormalizing a quantum field theory, we know that dimensional reduction is scheme dependent [4] [23], being optimal in BPHZ subtractions at zero momenta and temperature β^{-1}. We are working with a modified minimal subtraction scheme at zero momenta and temperature β^{-1}, similarly to what has been done by Farakos et al [24]. Now we are ready to find the exact form of the couterterms and then to compute the thermal renormalization group functions. Before proceeding, however, let us carefully examine the above expression. We notice that the form of the divergences ($s = 1, 2$) are of the same type of the ultraviolet divergences of a theory in four dimensions. The role of the thermal counterterms we want to choose is exactly to cancel out those divergences to give a theory well behaved in the high temperature limit. To find the exact form of these counterterms we have to impose renormalization conditions to correlation functions of the reduced theory. As we know, the ultraviolet divergences of a thermal quantum field theory are the same of the equivalent zero temperature model, that is, the difference of imposing conditions at zero temperature and temperature T is a finite renormalization (which is essential for dimensional reduction to occur). Keeping this in mind we impose conditions consistent with the tree level

action of the effective three dimensional theory. By requiring

$$\frac{\partial^2 V_{\text{eff}}}{\partial \phi_0^2}(\beta,\phi_0)\bigg|_{\phi_0=0} = -\mu^2 + \delta\mu^2 = \mu_R^2, \tag{24}$$

and

$$\frac{\partial^4 V_{\text{eff}}}{\partial \phi_0^4}(\beta,\phi_0)\bigg|_{\phi_0=0} = 6(\lambda+\delta\lambda)T = 6\lambda_R T, \tag{25}$$

we get

$$\delta\mu^2 = (N+2)\left[\lambda\frac{T^2}{24} - \lambda\frac{(-\mu^2)}{16\pi^2}\left[\frac{1}{\varepsilon} + \Psi(2) - 4\sum_{n=1}^{\infty}\ln\left(\frac{n\beta}{2}(-\mu^2)^{\frac{1}{2}}\right) + \gamma\right]\right], \tag{26}$$

and

$$\delta\lambda = -\frac{(N+8)}{6}\left[\frac{\lambda^2}{64\pi^2}\left[\frac{1}{\varepsilon} + \Psi(1) - 4\sum_{n=1}^{\infty}\ln\left(\frac{n\beta}{2}(-\mu^2)^{\frac{1}{2}}\right) + \gamma\right]\right], \tag{27}$$

where $\Psi(z)$ is the usual Psi function and γ is the Euler number $\Psi(1) = -\gamma$ [22]. Choosing those quantities as mass and coupling constant counterterms we guarantee that the new effective action for the reduced theory is finite for very high, but still finite, temperatures. However in the effective potential given by eq.(23), there is a new contribution, independent of T, coming from the evaluation of the six-point function. The computation of the six-point function with non-static internal loops gives a contribution for the static effective potential, as expected since we do not have obtained a perfect decoupling of non-static modes, but also gives an extra interaction term which was not present in the original action

$$\frac{\partial^6 V_{\text{eff}}}{\partial \phi_0^6}(\beta,\phi_0)\bigg|_{\phi_0=0} = ... + \frac{(N+26)}{9}\frac{\lambda^3\zeta(3)}{2^9\pi^4}, \tag{28}$$

exactly in the same sense as in the case of the scalar ϕ^4 theory [6]. We also see that, as expected, the non-renormalizable interaction terms are suppressed by negative powers of T in the limit of high temperatures. Indeed, for $s > 2$ the sum over n is analytic and is expressed in terms of usual zeta functions. Hence we are left, as we can see from eq.(23), with $T^s(\frac{\beta}{2\pi})^{2s-3} \sim T^{3-s} = T^{3-\frac{N}{2}}$ as coeficient of the sum over n. From this behavior we can check the temperature dependence of the 2, 4 and 6-point functions computed and neglect all the $N > 6$ contributions.

Finally, the effective three dimensional reduced theory has the following action

$$\Gamma(\phi_0) = \int d^3\mathbf{x} \left\{ \frac{1}{2}(\partial_i \phi_0)^2 + \frac{1}{2}\mu_R^2 \phi_0^2 + \frac{\lambda_R T}{4}\phi_0^4 + \frac{(N+26)}{9}\frac{\lambda_R^3 \zeta(3)}{2^9 \pi^4}\phi_0^6, \right\}. \tag{29}$$

We see from eq.(29) that, as long as T is finite, the reduced model defines a well behaved theory at the tree level. In the limit of infinite temperatures the "new" bare coupling constant $\lambda_R T$ imposes a second "renormalization" procedure. This action has the form of the standard action for a massive scalar field with φ^4 and φ^6 self-interaction terms

$$\Gamma(\varphi) = \int d^3\mathbf{x} \left(\frac{1}{2}(\partial_i \varphi)^2 + \frac{a}{2}\varphi^2 + \frac{b}{4}\varphi^4 + \frac{c}{6}\varphi^6 \right), \tag{30}$$

in a three-dimensional space. The study of the criticality of such a system can be found in a great variety of text books (see for instance [26]) and a general discussion can help us to understand the consequences of dimensionally reducing a quantum field theory which has a symmetry-restoring phase, as in our case. In the study of mean-field theory for ferromagnetic systems of the type given by eq. (30) we have situations in which the parameters a and b (c is fixed in order to have a potential bounded from below) can be made to vanish. This occurs for instance in $He^3 - He^4$

mixtures or some metamagnetic systems. Depending on the values of a and b, we can have first-order phase transitions, second-order phase transitions or both (the tricritical phenomenon [25]). If we carefully examine eq.(29) we see that one of the relevant parameters, the renormalized thermal coupling λ_R defined in eq.(25), is positive, at least for small N, since the bare quantity λ is also positive by definition. Thus we do not have a change in sign for both φ^4 and φ^6 terms in eq.(30) and the reduced theory has a stable vacuum. On the other hand, if we look for the mass term in the reduced action of eq.(29) we see that in the limit of high temperatures, its thermal correction causes a change in sign and the final expression is exactly the one in eq.(30) with all positive coeficients, i.e. we do not have the possibility of phase transitions of any kind. The conclusion is that in the limit of high temperatures the symmetry-restoring phase of the former model is unambigously mapped in the reduced one. Finally, to consider all the possibilities, we still have to argue what happens in the limit $N \to \infty$. In this case, the renormalized coupling λ_R has its sign changed for some suficiently large N, as we can see from eq.(27), and, in this limit, the reduced model is the one of a unstable theory. This result has been obtained by Linde [27] and more recently by Da Silva [28].

For completeness we finally compute the thermal renormalization group functions. As previously said in the introduction, the transition from a four-dimensional theory to an effective three-dimensional one is renormalization point dependent. This is not surprising since the Appelquist-Carazone decoupling theorem only holds if a particular class of renormalization prescriptions is adopted. Neverthless we can write down renormalization group equations. The independence of

bare correlation functions on the choice of the renormalization point, expressed by the conditions

$$\mu \frac{d}{d\mu}\Gamma^{(N)} = 0, \tag{31}$$

$$T \frac{d}{dT}\Gamma^{(N)} = 0, \tag{32}$$

give us those equations. Since we are only interested on the computation of thermal renormalization group functions, we will consider solely

$$\left(T\frac{\partial}{\partial T} + \beta_T \frac{\partial}{\partial \lambda} + \gamma_T m^2 \frac{\partial}{\partial m^2} \right) \Gamma^{(N)} = 0 \tag{33}$$

where

$$\beta_T = T\frac{\partial \lambda}{\partial T}, \tag{34}$$

$$\gamma_T = m^{-2} T \frac{\partial m^2}{\partial T}, \tag{35}$$

and

$$\Gamma^{(N)} = \left. \frac{\partial^N V_{\text{eff}}}{\partial \phi_0^N} \right|_{\phi_0 = 0} . \tag{36}$$

Using the results of the previous section we easily get

$$\beta_T = \frac{(N+8)}{6} \frac{\lambda^2}{16\pi^2} \tag{37}$$

and

$$\gamma_T = (N+2) \left[\frac{\lambda T^2}{12 m^2} + \frac{\lambda}{16\pi^2} \right], \tag{38}$$

as reasonably expected. If we now solve the eqs.(37) and (38) we obtain the renormalized parameters m and λ as functions of T, i.e. the thermal renormalized running mass and coupling constant.

4 Conclusions

In this paper the dimensional reduction of the $O(N)$ invariant linear σ model is performed. The effective potential for the reduced theory is exhibited in two different ways. From its exact form we can see that the ultraviolet divergences for the reduced theory are of the same kind of the ones for the original theory. On the other side, from its series expansion on the static fields we easily obtain the thermal counterterms that allow dimensional reduction to take place. Moreover, a new renormalizable interaction term, for the reduced theory, came up without changing the critical behavior of the former one. Some of the main features of dimensional reduction are present, as the fact that the effective action is a polinomial on the field because nonrenormalizable interaction terms are suppressed by powers of β in the limit of high temperatures ($\beta \to 0$). In addition, we also find, in the reduced model, a characteristic already observed by Linde [27], in another context, in which the potential becomes unstable for large values of N. A natural continuation of this paper would be to perform the dimensional reduction of the non-linear σ model. It also might be interesting to ask what happens to the tree-dimensional eletrodynamics with a Maxwell-Chern-Simons term in the high-temperature regime after integration over both fermionic and non-zero bosonic Matsubara modes. Note that the inclusion, in the Lagrangian, of the Chern-Simons term

avoids infrared divergences. These subjects are already under investigation.

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